Chapter 5 and 6

Section 4: Independence, Basis, and Dimension

(book sections 5.2, 6.3 and 6.4)

Ideas in this section…

- Every vector space / subspace is the span of some collection of vectors (we'll focus on the ones where the spanning set is finite)
- Given a vector space, we want to find the least amount of vectors possible that will span it
- This smallest number is unique and is called the dimension of the vector space
- A minimal collection of vectors that spans a vector space is called a basis of the vector space
- If a collection of vectors spans a subspace U, we can narrow it down to a basis of U
- If a collection of vectors in a subspace U is linearly independent, we can extend it to a basis of U

Finding a Spanning Set of V Discussion

Discussion

Q1: Is every vector space spanned by a finite set of vectors?

$\mathbb{R}^n = span\{\, \vec{e}_1 , \vec{e}_2 , \dots , \vec{e}_n \,\}$

M_{mn} is spanned by all matrices that have a 1 as one of the entries and a 0 for all other entries

$P_n = span\{\, 1 , \; x , \; x^2 , \; x^3 , \; \dots , \; x^n \,\}$

P is not spanned by any finite set of vectors

$F(-\infty, \infty)$ is not spanned by any finite set of vectors

A1: NO! But in this section, we are going to focus on vectors spaces that ARE.

Finding a Spanning Set of V Discussion

Discussion

Q2: If a vector space is spanned by a finite set of vectors, how can you find a spanning set?

A2:

- Start with any vector and look at its span
- If you end up with the vector space V, you're done
- If not, add in an additional vector and look at the span of the new set of vectors
- Eventually you will end up with a set of vectors that spans V as long as you keep adding in vectors that are not already in the span of the previous vectors

  Go over. Why for last bullet point comes later.

Finding a Spanning Set of V Discussion

Results:

1) $span\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subseteq span\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{v}_{n+1}\}$

2) $span\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = span\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{v}_{n+1}\}$ iff $\vec{v}_{n+1}$ IS a linear combination of the others vectors

3) $span\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subsetneq span\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{v}_{n+1}\}$ iff $\vec{v}_{n+1}$ IS NOT a linear combination of the others vectors

After the proofs, go over the idea of minimal spanning set and independence

Finding a Spanning Set of V Discussion

Results:

1) $span\{\,\vec{v}_1\,,\vec{v}_2\,,\ldots,\vec{v}_n\,\} \subseteq span\{\,\vec{v}_1\,,\vec{v}_2\,,\ldots,\vec{v}_n\,,\vec{v}_{n+1}\,\}$

Proof:

Finding a Spanning Set of V Discussion

Results:

2) $span\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = span\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{v}_{n+1}\}$ iff $\vec{v}_{n+1}$ IS a linear combination of the others vectors

Proof:

Finding a Spanning Set of V Discussion

Results:

3) $span\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subsetneq span\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{v}_{n+1}\}$ iff $\vec{v}_{n+1}$ IS NOT a linear combination of the others vectors

Proof:

Independence

<u>Def</u>: A set of vectors $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}$ in a vector space *V* is called <u>linearly independent</u> if the only linear combination of them that results in the zero vector must have all of its coefficients equal to 0. That is…

If $\exists \ c_1, c_2, \ldots c_n \in \mathbb{R}$ such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$, then

$$c_1 = c_2 = \cdots = c_n = 0.$$

Otherwise, if $\exists \ c_1, c_2, \ldots c_n \in \mathbb{R}$ that are not all 0, such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$, then the vectors $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}$ are <u>linearly dependent</u>, or <u>not linearly independent</u>.

Independence

If the vectors $\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$ are linearly independent, then the following are equivalent …

1) Def: If $\exists\ c_1, c_2, \dots c_n \in \mathbb{R}$ such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$, then

$$c_1 = c_2 = \cdots = c_n = 0.$$

(This is what we will use when showing that a set of vectors IS linearly independent)

2) None of these vectors is a linear combination of the others

3) Different linear combinations of these vectors produce different vectors

(Or equivalently from the book: Each vector in $span\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$ has a unique representation as a linear combination of the vectors in $\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$)

Independence

If the vectors $\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$ are linearly independent, then the following are equivalent …

1) Def: If $\exists\ c_1, c_2, \dots c_n \in \mathbb{R}$ such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$, then

$$c_1 = c_2 = \cdots = c_n = 0.$$

2) None of these vectors is a linear combination of the others

Proof 1→2:

Independence

If the vectors $\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$ are linearly independent, then the following are equivalent …

1) Def: If $\exists \; c_1, c_2, \dots c_n \in \mathbb{R}$ such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$, then
$$c_1 = c_2 = \cdots = c_n = 0.$$

2) None of these vectors is a linear combination of the others

Proof 2→1:

Independence

If the vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are linearly independent, then the following are equivalent …

1) Def: If $\exists\ c_1, c_2, \ldots c_n \in \mathbb{R}$ such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$, then
$$c_1 = c_2 = \cdots = c_n = 0.$$

3) Different linear combinations of these vectors produce different vectors

Proof 1→3:

Independence

If the vectors $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}$ are linearly independent, then the following are equivalent …

1) Def: If $\exists\ c_1, c_2, \ldots c_n \in \mathbb{R}$ such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$, then

$$c_1 = c_2 = \cdots = c_n = 0.$$

3) Different linear combinations of these vectors produce different vectors

Proof 3→1:

Independence

To prove that the vectors $\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$ are linearly independent…

1) Set up the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$
2) Solve for the constants $c_1, c_2, \dots, c_n$
3) If the only solution is that all the constants are 0, then the vectors are linearly independent

To prove that the vectors $\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$ are linearly dependent…

1) Find constants $c_1, c_2, \dots, c_n$ that are not all 0 such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

Independence

Ex 1a: Are the following vectors linearly independent?

a) $(1,2,3,4)$, $(2,5,0,-1)$, $(1,1,3,0)$

Note About Systems of Linear Equations and Matrix Equations

$$\begin{aligned} ax_1 + bx_2 &= e \\ cx_1 + dx_2 &= f \end{aligned} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \qquad x_1\begin{bmatrix} a \\ c \end{bmatrix} + x_2\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

<u>Are all equivalent!</u>

Independence

<u>Thm 5.2.2 (part 1)</u>:
Let $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n$ be vectors in $\mathbb{R}^m$ and let $A = [\vec{c}_1\ \vec{c}_2\ \dots\ \vec{c}_n]$.
Then $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n$ are linearly independent iff the system of equations $A\vec{x} = \vec{0}$ only has the trivial solution $\vec{x} = \vec{0} \in \mathbb{R}^n$.

I.e. To see if a bunch of vectors in $\mathbb{R}^m$ are linearly independent,

1) make these vectors the columns of a matrix A
2) then solve the system $A\vec{x} = \vec{0}$
3) row reduce the augmented matrix $[A|\vec{0}]$
4) the vectors are linearly independent if there is a leading 1 in every column (on the left of the augmentation line). Otherwise, they aren't.

Independence

<u>Ex 1b</u>: Are the following vectors linearly independent?

b) $(2, 4, 0, -5)$, $(3, -4, 2, 2)$, $(1, 12, -2, -12)$

Independence

<u>Ex 2a</u>: Show that $\{\, 1 + x,\ \ 3x + x^2,\ \ 2 + x - x^2\}$ is a linearly independent set of vectors in P_2

Independence

Ex 2b: Show that $\{1 + 3x + x^2,\ 5 - 2x^2,\ 3 + x,\ 1\}$ is a linearly dependent set in P_2

Independence

Ex 3: Show that $\left\{ \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 0 & 10 \end{bmatrix} \right\}$ is a linearly dependent set in M_{22}

Independence

<u>Ex 4a</u>: Show that $\{\sin x, \cos x\}$ is a linearly independent set of vectors in $F[0,2\pi]$ of all functions defined on $[0,2\pi]$

Independence

Ex 4b: Show that $\{\sin^2 x, \cos^2 x, 1\}$ is a linearly dependent set of vectors in $F[0,2\pi]$ of all functions defined on $[0,2\pi]$

Independence

Ex 5: Show that $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a linearly independent set of vectors in $\mathbb{R}^n$

Independence

<u>Ex 6</u>: Show that $\vec{0}$ cannot belong to any linearly independent set of vectors.

Basis and Dimension

<u>Def</u>: A set of vectors $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}$ in a vector space V is a <u>basis</u> for V if…

1) $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}$ is linearly independent and
2) $\text{span}\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \} = V$

Basis and Dimension

Thm 6.3.2 (Fundamental Theorem):

If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ is a finite linearly independent set of vectors of V and $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is a finite set of vectors that spans V then $m \leq n$.

Proof:

Basis and Dimension

Thm 6.3.2 (Fundamental Theorem):

If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ is a finite linearly independent set of vectors of V and $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is a finite set of vectors that spans V then $m \leq n$.

Proof (continued):

Basis and Dimension

Theorem 6.3.3: Invariance Theorem

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ *and* $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ *be two bases of a vector space* V. *Then* $n = m$.

Proof:

Basis and Dimension

Def: If $\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$ is a basis for a vector space V, the number n of vectors in the basis is called the dimension of the vector space V.

Notation: $\dim V = n$

Why does this definition make sense?

Note: The zero vector space $\{ \vec{0} \}$ is defined to have dimension 0 and a basis for the zero vector space $\{ \vec{0} \}$ is defined to be the empty set $\emptyset$

Basis and Dimension

Ex 7a: Because $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is linearly independent and spans $\mathbb{R}^n$, $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$ (called the standard basis for $\mathbb{R}^n$) and $\dim \mathbb{R}^n = n$

Ex 7b: The set of all matrices that have a 1 as one of the entries and a 0 for all other entries is a basis for M_{mn} (this is the standard basis for M_{mn}) and $\dim M_{mn} = mn$

Ex 7c: The standard basis for P_n is $\{1, x, x^2, x^3, \ldots, x^n\}$ and $\dim P_n = n + 1$

Basis and Dimension

<u>Results</u>: Suppose vector space V has finite dimension n.

1) Any linearly independent set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ $(k \leq n)$ can be enlarged to a basis of V Discussion / Proof

Basis and Dimension

<u>Results</u>: Suppose vector space V has finite dimension n.

2) Any finite spanning set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ $(k \geq n)$ can be reduced to a basis of V Discussion / Proof

Basis and Dimension

Results: Suppose vector space V has finite dimension n.

3) $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent iff $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ spans V

Discussion emphasize number of vectors / Proof

Basis for $\mathbb{R}^n$ Theorem

Theorem 5.2.3

The following are equivalent for an $n \times n$ matrix A

1. *A is invertible.*
2. *The columns of A are linearly independent.*
3. *The columns of A span $\mathbb{R}^n$.*
4. *The rows of A are linearly independent.*
5. *The rows of A span the set of all $1 \times n$ rows.*

Discussion / Proof

Basis and Dimension

Ex 8: Show that $(1,4,1)$, $(9,-2,-3)$, $(-3,-1,0)$ are linearly independent and span $\mathbb{R}^3$. $\det = 18$

Enlarging a Set to a Basis

Ex 9: Extend $\{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = (1,4,0,-1)$ and $\vec{v}_2 = (2,1,3,1)$ to a basis of $\mathbb{R}^4$.

Enlarging a Set to a Basis

Ex 10: Enlarge the independent set $D = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ to a basis of M_{22}.

Enlarging a Set to a Basis

Ex 11: Find a basis of P_3 containing the independent set $\{1+x,\ 1+x^2\}$.

Reducing a Set to a Basis

Ex 12: Find a basis and calculate the dimension of

$$span\{\ (-1,2,1,0),(2,0,3,-1),\ (4,4,11,-3),(3,-2,2,-1)\ \}$$

Defer to Section 5.4 Lecture

Reducing a Set to a Basis

Ex 13: Find a basis of P_3 in the spanning set

$$\{1,\ x + x^2,\ 2x - 3x^2,\ 1 + 3x - 2x^2,\ x^3\}$$

Defer to Section 5.4 Lecture

Basis and Dimension of Subspaces

Ex 14: Find a basis and calculate the dimension of

$$U = \left\{ \begin{bmatrix} a \\ a+3b \\ a-b \\ 2b \end{bmatrix} \middle| \; a, b \in \mathbb{R} \right\}$$

Basis and Dimension of Subspaces

<u>Ex 15</u>: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and consider the subspace

$$U = \{\, X \in M_{22} \mid AX = XA \,\}.$$

Show that $\dim U = 2$ and find a basis for U.

Basis and Dimension of Subspaces

Ex 16: Show that the set V of all symmetric 2×2 matrices is a vector space and find its dimension.

Basis and Dimension of Subspaces

Ex 17: Let V be the space of all 2×2 symmetric matrices. Find a basis of V consisting of invertible matrices.

Basis and Dimension

<u>Result</u>: Let V be a vector space of dimension n and let U be a subspace of V. Then…

1) $\dim U \leq n$
2) If $\dim U = n$, then $U = V$.

<u>Proof</u>:

What you need to know from the book

Book reading

Section 5.2 pages 271 - 279
Section 6.3 pages 345 – 351
Section 6.4 pages 354 – 360

Problems you need to know how to do from the book

Section 5.2 page 280 #'s 1 - 20
Section 6.3 page 351 #'s 1 - 37
Section 6.4 page 361 #'s 1 - 26